STABILITY OF MOTION OF THE PLANE BOUNDARY SEPARATING TWO FLUIDS

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The effect of viscosity on the disintegration of liquid jets can be considered in two ways. First, viscous forces alter the basic flow: they form a boundary layer whose presence necessarily alters wave formation. Second, viscous forces can have a direct effect on the development of perturbations for a given velocity profile of the basic flow. In this case the study of stability must be based on the Navier-Stokes equations instead of the equations of an ideal fluid. This complicates analysis considerably. Available data [1] indicate that this influence is very minor in the case of moderately viscous fluids. It appears, therefore, that the principal role is played by changes in the velocity profile alone, and that the behavior of the perturbations is described by the equations of an ideal fluid.

In the present study we investigated the stability of motion and wave formation at the boundary between two fluids in order to determine the effect of viscosity on the drop formation mechanism. The simplest case of oscillation of the boundary is chosen in order to keep the analysis as simple as possible.

1. Let us consider the flow of three fluids of densities ρ_1, ρ_2, ρ_3 separated by parallel boundaries (e.g. water, water vapor, air).

We assume that the space is divided into four zones: the first and second zones are occupied by the flowing fluid of density ρ_1 which forms a boundary layer of thickness h_1 (the second zone); the third zone is occupied by the flowing liquid of density ρ_2 forming a boundary layer of thickness h_2; the fourth zone contains the fluid of density ρ_3, which is at rest (Fig. 1).

Surface tension forces act at the boundary between the second and third zones (the coefficient of surface tension is σ); the velocity in the boundary layer varies linearly with respect to the coordinate. The data for each zone (in accordance with Fig. 1) are as follows:

$$
\begin{array}{lll}
(1) & \rho_1 & V_1 = V, \quad (-\infty < y < -h_1) \\
(2) & \rho_1 & V_2 = V - (V - V_0)(h_1 + y)/h_1, \quad (-h_1 \leqslant y \leqslant 0) \\
(3) & \rho_2 & V_3 = V_0 - V_0 y / h_2, \quad (0 \leqslant y \leqslant h_2) \\
(4) & \rho_3 & V_4 = 0 \quad (h_2 \leqslant y < +\infty).
\end{array}
$$

We take the stream function in the form

$$\psi = \varphi(y) e^{i(\alpha x - \beta t)}. \tag{1.1}$$

The solutions for the function $\varphi(y)$ in zones 1, 2, 3, 4 are, respectively,

$$\varphi_1 = C_1 e^{\alpha y}, \quad \varphi_2 = C_2 e^{-\alpha y} + C_3 e^{\alpha y},$$
$$\varphi_3 = C_4 e^{-\alpha y} + C_5 e^{\alpha y}, \quad \varphi_4 = C_6 e^{-\alpha y}. \tag{1.2}$$

The equation of motion yields an expression for the pressure gradient,

$$\frac{\partial p}{\partial x} = -\rho\left(\frac{\partial u'}{\partial t} + V\frac{\partial u'}{\partial x} + v'\frac{dV}{dy}\right)$$

$$\left(u' = \frac{\partial \psi}{\partial y}, \quad v' = -\frac{\partial \psi}{\partial x}\right). \tag{1.3}$$

By (1.2) and (1.3),

$$u' = \varphi'(y) e^{i(\alpha x - \beta t)}, \qquad v' = -i\alpha\varphi(y) e^{i(\alpha x - \beta t)}. \tag{1.4}$$

Hence,

$$\frac{\partial p}{\partial x} = i\rho\left[(\beta - \alpha V)\varphi' + \alpha\varphi\frac{dV}{dy}\right] e^{i(\alpha x - \beta t)}. \tag{1.5}$$

Further, the rise of a fluid particle above the unperturbed boundary surface is given by

$$\eta = \eta^\circ e^{i(\alpha x - \beta t)},$$
$$\frac{\partial \eta}{\partial t} = v' - \frac{\partial \eta}{\partial x} V = -i\beta\eta^\circ e^{i(\alpha x - \beta t)} =$$
$$= -i\alpha\varphi e^{i(\alpha x - \beta t)} - i\alpha\eta^\circ V e^{i(\alpha x - \beta t)}. \tag{1.6}$$

Hence,

$$\eta^\circ = \frac{\alpha\varphi}{\beta - \alpha V}. \tag{1.7}$$

The normal velocity components and the pressure gradient at the boundaries $y = -h_1$, $y = 0$, $y = h_2$ are continuous, so that

$$
\begin{array}{ll}
\varphi_1 = \varphi_2, \quad \dfrac{\partial p_1}{\partial x} = \dfrac{\partial p_2}{\partial x} & \text{for } y = -h_1 \\
\varphi_2 = \varphi_3, \quad \dfrac{\partial p_2}{\partial x} - \dfrac{\partial p_3}{\partial x} = -\sigma\dfrac{\partial^3 \eta}{\partial x^3} = i\sigma\eta\alpha^3 & \text{for } y = 0 \\
\varphi_3 = \varphi_4, \quad \dfrac{\partial p_3}{\partial x} = \dfrac{\partial p_4}{\partial x} & \text{for } y = h_2.
\end{array}
\tag{1.8}
$$

Substituting (1.2), (1.5), (1.6), and (1.7) into (1.8) and introducing the dimensionless quantities

$$m_1 = \alpha h_1, \qquad m_2 = \alpha h_2,$$
$$M = \rho_3 / \rho_1, \qquad N = \rho_3 / \rho_2, \tag{1.9}$$

we obtain a system of six equations linear in the arbitrary constants. Eliminating arbitrary constants from the equations of this system, we obtain a characteristic equation of the form

$$
\begin{aligned}
&-[2\eta + g_1(1 - e^{-2m_1})] \times \\
&\times \{2M\eta_0^2[(1 - N)\beta + g_2]e^{-2m_2} - M(\eta_0^2 + \\
&+ g_2\eta_0)[\beta((1 + N) + (1 - N)e^{-2m_2}) - \\
&- g_2(1 - e^{-2m_2})]\} + \{\beta[(1 + N) + \\
&+ (1 - N)e^{-2m_2}] - g_2(1 - e^{-2m_2})\}\{[2\eta + g_1(1 - \\
&- e^{-2m_1})][\eta_0^2 - g_1\eta_0 - \tau] + 2\eta_0^2 g_1 e^{-2m_1}\} = 0,
\end{aligned}
$$
$$\eta = \beta - \alpha V, \quad \eta_0 = \beta - \alpha V_0, \quad g_1 = \frac{V - V_0}{h_1},$$
$$g_2 = \frac{V_0}{h_2}, \quad \tau = \frac{\sigma\alpha^3}{\rho_1}. \tag{1.10}$$

2. Let us consider the special case where the densities in zones 3 and 4 are equal (e. g., in the case of

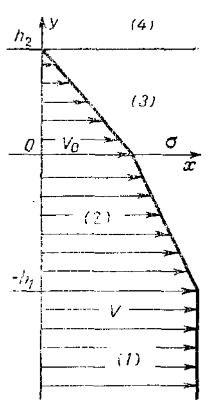

Fig. 1

the water–water vapor–air system, when air of the same density as that in zone 4 moves instead of vapor in zone 3, so that $\rho_2 = \rho_3$, N = 1). Instead of Eq. (1.10) we have

$$-[2\eta + g_1(1 - e^{-2m_1})]\{2M\eta_0^2 g_2 e^{-2m_2} - -M(\eta_0^2 + g_2\eta_0)[2\beta - g_2(1 - - e^{-2m_2})]\} + \{2\beta - g_2(1 - e^{-2m_2})\} \times \times\{[2\eta + g_1(1 - e^{-2m_1})][\eta_0^2 - g_1\eta_0 - - \tau] + 2\eta_0^2 g_1 e^{-2m_1}\} = 0. \qquad (2.1)$$

Equation (2.1) therefore corresponds to the problem of stability of the boundary separating two fluids of densities ρ_1 and ρ_2 (e.g., water and air) with a boundary layer of thickness h_1 in the first fluid and a boundary layer of thickness h_2 in the second. Let us set

$$h_1 = h \quad (m_1 = m), \qquad h_2 = kh \quad (m_2 = km),$$
$$K = \frac{1}{k+1}, \qquad V_0 = KkV. \qquad (2.2)$$

Then

$$\frac{\eta h}{V} = H - m, \qquad \frac{\eta_0 h}{V} = H - Kkm = R,$$
$$H = \frac{\beta h}{V}, \quad \frac{g_1 h}{V} = \frac{g_2 h}{V} = K,$$
$$\frac{\tau h^2}{V^2} = \frac{Mm^3}{W} = D, \qquad W = \frac{\rho_1 h V^2}{\sigma}. \qquad (2.3)$$

When we substitute (2.2) and (2.3) into Eq. (2.1), the latter becomes

$$(1 + M)R^4 + [(p + q)(1 + M) - -K(1 - M) + a - bM]R^3 + [pq(1 + M) - -(p + q)K(1 - M) + aq - bpM - D]R^2 - -[pqK(1 - M) + (p + q)D]R - pqD = 0,$$
$$p = -Km + {}^1/_2 K(1 - e^{-2m}),$$
$$q = Kkm - {}^1/_2 K(1 - e^{-2km}),$$
$$a = Ke^{-2m}, \quad b = Ke^{-2km}. \qquad (2.4)$$

Let us consider the special case in which there is no boundary layer in zone 3, i.e., the problem of oscillation of the boundary between two fluids (e.g., water and air) with allowance for a boundary layer in only one of them (in the water).

Setting k = 0 in (2.4), we find that

$$H^3 + p_2 H^2 + p_1 H + p_0 = 0$$
$$p_2 = -\frac{n_1 + Mn_2}{1 + M}, \quad p_1 = \frac{n_2 - D}{1 + M},$$
$$p_0 = \frac{n_2 D}{1 + M}, \quad \begin{matrix} n_1 = m + A \\ n_2 = m - A \end{matrix}, \quad A = \frac{1 - e^{-2m}}{2}. \qquad (2.5)$$

In the range of values of the parameters entering into its coefficients Eq. (2.5) has one real root $H_1 = a$ and two complex conjugate roots $H_{2,3} = b \pm ic$. The relationships between the roots and coefficients of this equation are given by the expressions

$$a + 2b = -p_2, \quad 2ab + b^2 + c^2 = p_1,$$
$$a(b^2 + c^2) = -p_0. \qquad (2.6)$$

To obtain an equation for determining the oscillation increment c we eliminate the quantities a and b from (2.6). This yields

$$\psi(\psi + 1)^2 = F, \quad c^2 = {}^1/_4 (p_2^2 - 3p_1)\psi,$$
$$F = \frac{27p_0^2 + 4p_1^3 + 4p_0p_2^3 - 18p_0p_1p_2 - p_1^2p_2^2}{(p_2^2 - 3p_1)^3}. \qquad (2.7)$$

Equation (2.7) should be used to determine the real roots only. It is easy to solve graphically by means of the curves F = F(m, W) and F = $\psi(\psi + 1)^2$ (the dashed curve in Fig. 2).

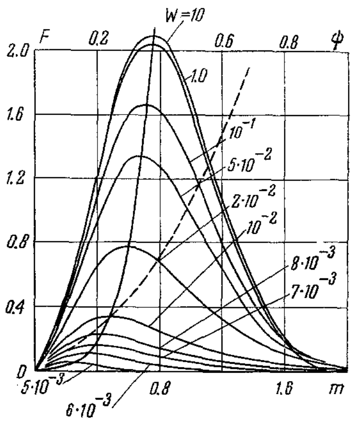

Fig. 2

Solutions of Eq. (2.7) appear in Figs. 3 and 4, which show the square H_i^2 of the dimensionless increment, as a function of the dimensionless wave number for several values of W, and the optimum wave number m_0 as a function of the Weber number (for M = $1.2\cdot10^{-3}$, water–air).

Let us consider some special cases.

We assume that the thickness h of the boundary layer tends to zero. Equation (2.5) then yields

$$\beta_i = \sqrt{MV^2\alpha^2 - \alpha^3\sigma/\rho_1}. \qquad (2.8)$$

This result was obtained by the authors of [2]. In the absence of a velocity or a second fluid (V = 0 or M = = 0), the oscillation increment β_i turns out to be imaginary. This indicates that the motion in this case is stable.

Squaring (2.8), differentiating the expression for the square of the increment with respect to zero, and equating the derivative to zero, we obtain an expression for the wavelength of the optimal perturbation (corresponding to the maximum of the oscillation increment),

$$\lambda_m = \frac{3\pi\sigma}{\rho_2 V^2}. \qquad (2.9)$$

In the other limiting case (W = ∞), setting D = 0 in the solution of Eq. (2.5), we obtain the optimal wavenumber m corresponding to the asymptote in Fig. 4,

$$m_\infty = 1.225. \qquad (2.10)$$

From this we obtain the limiting wavelength in the presence of a boundary layer in the liquid for W → ∞ (practically speaking, for W > 0.1),

$$\lambda_m \geqslant \frac{2\pi h}{1.225} = 5.12h. \qquad (2.11)$$

Hence, in this limiting case of a boundary layer present in the liquid the wavelength cannot be smaller than approximately five times the thickness of the boundary layer for any value of the liquid velocity.

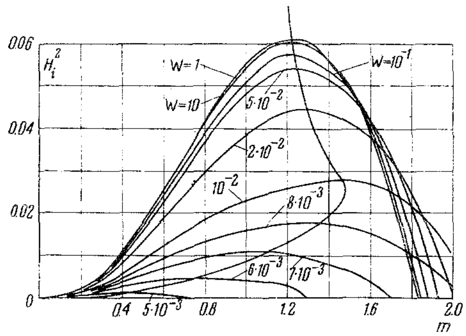

Fig. 3

As we see from Fig. 4, the value of the dimensionless wave number in the general case does not exceed m_0 = 1.5, which corresponds to a wavelength of $\lambda_m \geq$ ≥ 4.2h. From the same figure we see that the boundary surface becomes stable for Weber numbers W ≤ ≤ 0.004.

The above problem of oscillation of the boundary between two fluids with allowance for the boundary layer in one of them (in the denser fluid) is the most important one, since the boundary layer in the less dense fluid has only a weak effect on the oscillations.

This may be seen from the following considerations. From Eq. (2.4) for D = 0 (capillarity is not involved) we can obtain simple equations corresponding to:

1. The case of oscillations of the boundary between two fluids with allowance for the boundary layer in the denser fluid only,

$$(M+1)H^2 - (Mn_2 + n_1)H + n_2 = 0.$$

2. The case of oscillations of the boundary between two fluids with allowance for the boundary layer in the less dense fluid only,

$$(M+1)H^2 - (n_2 + Mn_1)H + Mn_2 = 0.$$

Solution of each of the above equations simultaneously with transformed equation (2.8) corresponding to the case of oscillations of the

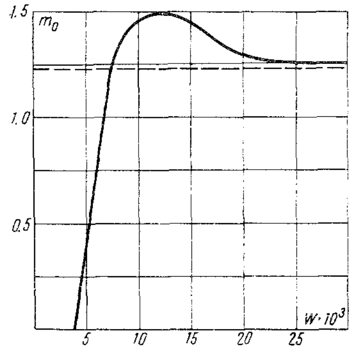

Fig. 4

boundary between two fluids with a discontinuous velocity distribution enabled us to draw the following conclusions:

(1) The presence of a boundary layer in the denser fluid markedly increases the oscillation increment as compared with the case of a discontinuous velocity distribution at the boundary (the lower-density fluid has only a weak effect).

(2) The presence of a boundary layer in the lower-density fluid (because of its low density) yields values of the oscillation increments comparable with those obtained in the case of a discontinuous velocity distribution.

3. If the thickness of the boundary layer in the liquid or gas is small, the viscosity of the liquid or gas can have a direct effect on wave formation. The problem of the direct effect of viscosity on wave formation without allowance for the velocity of the fluid was solved by Lamb [3] (pp. 787–791), who found that viscosity resulted in damping of the oscillations according to the law

$$A = A_0 e^{-2\nu\alpha^2 t}, \qquad \tau = \frac{\lambda^2}{8\pi^2\nu}. \qquad (3.1)$$

Here A is the wave amplitude, A_0 is the initial amplitude, ν is the kinematic viscosity of the liquid, α is

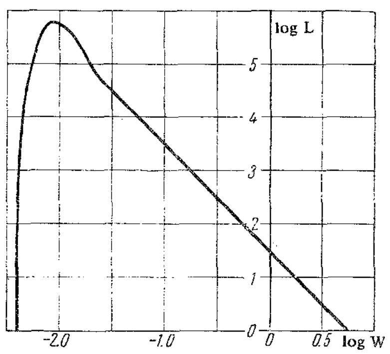

Fig. 5

the wave number, and τ is the time required for the amplitude to decrease e times (e is the base of Napierian logarithms).

This means that viscosity affects short waves only.

It is of interest to compare the direct effect of viscosity on oscillations with the effect by way of the boundary layer in the presence of a fluid velocity.

As in [4], we estimate the wavelength from the formula for the arbitrary amplitude A of the waves with respect to time [3]:

$$\frac{dA}{dt} = \frac{C}{2\rho_1 c} - 2\nu_1\alpha^2 A,$$

$$c = \sqrt{\sigma\alpha/\rho_1}, \qquad \nu_1 = \frac{\mu_1}{\rho_1}, \qquad \alpha = \frac{2\pi}{\lambda} \quad . \tag{3.2}$$

Here c is the velocity of propagation of the capillary waves, ν_1 is the coefficient of kinematic viscosity of the fluid, α is the wave number, and λ is the wavelength.

According to Jeffrey [3], the expression for the gas pressure over the moving wavecrest can be taken in the form

$$\beta^\circ\rho_2 (U_2 - c)^2 d\eta \,/\, dx,$$

where U_2 is the velocity of the gas, $\beta^0 \leq 1$ is a coefficient characterizing the gas pressure distribution over the wave crest, and $d\eta/dx$ is the derivative of the rise of the fluid surface with respect to the coordinate in the direction of the velocity U_2. Then

$$C = \beta^\circ\rho_2 (U_2 - c)^2 \alpha A. \tag{3.3}$$

Substituting the values of C and c into (3.3) and neglecting the wave propagation velocity as compared with U_2, we obtain

$$\frac{dA}{dt} = \frac{A}{\tau}, \qquad \left(\frac{1}{\tau} = \frac{\beta^\circ}{2}\frac{\rho_2}{\rho_1}\frac{U_2^2\alpha}{\sqrt{\sigma\alpha}}\sqrt{\rho_1} - \frac{2\mu_1\alpha^2}{\rho_1}\right). \tag{3.4}$$

The amplitude A is maximum for $1/\tau = 0$. From this we find that

$$W_\lambda = \frac{4\pi\sqrt{2\pi}}{\beta^\circ} L_\lambda^{-1/2} \qquad \left(W_\lambda = \frac{\rho_2 U_2^2 \lambda}{\sigma},\ L_\lambda = \frac{\lambda\sigma\rho_1}{\mu_1^2}\right). \tag{3.5}$$

Since the optimum wavelength is

$$\lambda = 2\pi h \,/\, m_0(W), \tag{3.6}$$

where $m_0(W)$ must be taken from the curve of Fig. 4, we can substitute (3.6) into (3.5) to obtain the following expression for the case of a boundary layer in the liquid only:

$$L = 16\, [m_0(W)]^3 \,/\, W^2. \tag{3.7}$$

Here we assumed that $\beta^\circ = 1/2$.

Equation (3.7) was derived from the condition of equality of the optimum perturbation wavelength obtained with allowance for the boundary layer but not for the viscosity, to the wavelength obtained with allowance for a velocity distribution discontinuous at the boundary surface.

This equation enables us to estimate the range of applicability of the above theory.

The curve of Fig. 5 constructed on the basis of Eq. (3.7) indicates that with low-viscosity liquids (water et al.) the effect of viscosity is negligible. For example, for a velocity of V = 100 m/sec and h = = 0.02 cm for water we have W = 37, L = 13 460; the point corresponding to these values of W and L lies high above the curve of Fig. 5.

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